

On the Number of Single-Peaked Narcissistic or Single-Crossing Narcissistic Preferences

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Abstract

We investigate preference profiles where voters have preference orders (i.e. linear orders) over a set of alternatives. Such a profile is *narcissistic* if each alternative is preferred the most by at least one voter. It is *single-peaked* if there is a linear order of the alternatives such that each voter's preferences over the alternatives along this order are either strictly increasing, or strictly decreasing, or first strictly increasing and then strictly decreasing. It is *single-crossing* if there is a linear order over the voters such that each pair of voters divides the order into at most two suborders, where in each suborder, all voters have the same linear order over this pair. We show that for n voters and n alternatives, the number of single-peaked narcissistic profiles is $\prod_{i=2}^{n-1} \binom{n-1}{i-1}$ while the number of single-crossing narcissistic profiles is $2^{\binom{n-1}{2}}$.

1 Introduction

When forming coalitions [11, 5], building teams [3], or playing games, the individuals, who we jointly denote as *voters*, may have preferences over who is better than another as a potential coalition partner, a team member, or a player. Thus, each voter has a preference order (which is a linear order) over the whole group of voters. Deriving from a simple psychological model, it seems natural to assume that each voter is *narcissistic* [3], meaning that she is her own ideal and, thus, most preferred *alternative*.

Another assumption on voter preferences over a set of alternatives, the *single-peaked* property, is characterized by a left-to-right order of the alternatives such that for each voter, her preferences along the order strictly increase until they reach the *peak* which is her most preferred alternative, and then strictly decrease. Black [4] introduced the concept of single-peakedness. He observed that voters' political interests over the parties are single-peaked, meaning that the parties can be ordered from left to right such that each voter has a political ideal on this order and the further a party is from this ideal, the less she will like this party. Single-peaked preferences are also studied in psychology under the name *unimodal orders* [10, 12].

A third assumption, the *single-crossing* property, requires that the voters can be linearly ordered in such a way that the preference orders of the voters over each pair

of alternatives along this order changes at most once. Mirrlees [18] and Roberts [19] introduced this concept. They observed that voters' preferences on income taxation display a pattern that accords to their incomes, and are thus single-crossing: When asked about the preferences over two tax rates x and y with $x > y$, if a voter v (the “crossing” spot) with medium income already changes from preferring x over y to preferring y over x , then all voters with higher income than v will also change to prefer y over x . Single-crossing preferences are also studied in applied mathematics under the name of weak Bruhat orders [17].

Research on restricted domains such as single-peaked or single-crossing preferences has been popular in political science, in psychology, in social choice, and quite recently in computational social choice. We refer to the papers of Brederick et al. [7], Elkind et al. [14] for ample references to research on the two properties. Single-crossing preferences are not necessarily single-peaked, but Saporiti and Tohmé [20] and Barberà and Moreno [2] observed that single-crossing narcissistic preferences are single-peaked. However, not all single-peaked narcissistic preferences are single-crossing. For a simple illustration, the preferences of the following four voters are narcissistic. For instance, voter v_1 ranks herself at the first position, and ranks v_2 , v_3 , and v_4 at the second, third, and fourth position, respectively. These voter preferences are single-crossing with respect to the order $v_1 \triangleright v_2 \triangleright v_3 \triangleright v_4$ and hence single-peaked (with respect to the same order). See Example 2 for more information.

$$\begin{aligned} v_1: v_1 &\succ_1 v_2 \succ_1 v_3 \succ_1 v_4, \\ v_2: v_2 &\succ_2 v_3 \succ_2 v_4 \succ_2 v_1, \\ v_3: v_3 &\succ_3 v_2 \succ_3 v_4 \succ_3 v_1, \\ v_4: v_4 &\succ_4 v_3 \succ_4 v_2 \succ_4 v_1. \end{aligned}$$

However, if we just swap the positions of v_4 and v_1 in the preference order of voter v_3 to obtain

$$v_3: v_3 \succ_3 v_2 \succ_3 v_1 \succ_3 v_4,$$

then the resulting voter preferences are still single-peaked and narcissistic but not single-crossing anymore. See Example 2 for more information.

In this work, we deal with preference profiles with n voters who each have a preference order over all n voters. In general, there are $n!^n$ different preference profiles. But how likely is it that one arbitrarily chosen of these profiles will have some specific property? For instance, the number of narcissistic profiles is $(n-1)!^n$. So, one out of n profiles is narcissistic. Bruner and Lackner [8] studied the likelihood of single-peaked preferences under some distribution assumption on the preference orders of the voters. However, we are interested in narcissistic profiles that are also single-peaked or single-crossing. More precisely, we investigate the number of narcissistic profiles that are also single-peaked (SPN) or single-crossing (SCN). While it is quite straight-forward to derive the number of SPN profiles, this is not the case for SCN profiles. Nevertheless, with the help of *semi-standard Young tableaux* (SSYT), in particular, by establishing a bijective relation between SSYTs and SCN profiles, we are able to determine the number of SCN profiles.

Our results are that for n voters and n alternatives, the number of single-peaked narcissistic profiles is $\prod_{i=2}^{n-1} \binom{n-1}{i-1}$ while the number of single-crossing narcissistic profiles is $2^{\binom{n-1}{2}}$.

2 Basic Definitions and Notation

In this section, we introduce basic terms from social choice, combinatorics of permutations, and Young tableaux.

2.1 Voters, alternatives, and preference orders

Let $\mathcal{V} := \{1, 2, \dots, n\}$ be a set of voters. Since we are concerned with voters that have preferences over themselves, \mathcal{V} also denotes the set of alternatives. A *preference order* \succ over \mathcal{V} is a strict linear order over \mathcal{V} , that is, a binary relation which is total, antisymmetric, and transitive. Sometimes, we use the letters a, b, c, \dots instead of the numbers $1, 2, \dots$ to emphasize that we are considering the alternatives instead of the voters. Given two disjoint subsets of alternatives A and B , we use the notation $A \succ B$ to express that a voter has a preference order \succ such that for each $a \in A$ and for each $b \in B$ it holds that $a \succ b$. We simplify $A \succ B$ to $a \succ B$ if $A = \{a\}$ and $A \succ B$ to $A \succ b$ if $B = \{b\}$.

A preference profile $\mathcal{P}(\mathcal{V})$ of voter set \mathcal{V} is a collection of preference orders for \mathcal{V} :

$$\mathcal{P}(\mathcal{V}) := (\succ_1, \succ_2, \dots, \succ_n),$$

where each \succ_i represents the preference order of voter i .

Example 1. If we rename the voters $v_i \mapsto i$ for all $i \in \{1, 2, 3, 4\}$ in the introductory example, then we obtain the following preference profile for the voter set $\{1, 2, 3, 4\}$:

voter 1: $1 \succ_1 2 \succ_1 3 \succ_1 4$,
voter 2: $2 \succ_2 3 \succ_2 4 \succ_2 1$,
voter 3: $3 \succ_3 2 \succ_3 4 \succ_3 1$,
voter 4: $4 \succ_4 3 \succ_4 2 \succ_4 1$.

To describe the properties of preference profiles, for each preference order \succ and each subset of alternatives $\mathcal{V}' \subseteq \mathcal{V}$, we introduce the concept of *top* alternatives i from \mathcal{V} that are preferred over \mathcal{V}' .

For each preference order \succ and

for each subset $\mathcal{V}' \subseteq \mathcal{V}$ of alternatives:

$$\text{top}(\succ, \mathcal{V}') := \{i \in \mathcal{V} \mid \forall j \in \mathcal{V}' \setminus \{i\} \text{ it holds that } i \succ j\}.$$

For example, the top alternatives of preference order \succ_2 with respect to $\{3, 4\}$ are 2 and 3. Thus, $\text{top}(\succ_2, \{3, 4\}) = \{2, 3\}$.

We use $\text{peak}(\succ)$ to denote the most preferred alternative in \succ , that is, $\{\text{peak}(\succ)\} := \text{top}(\succ, \mathcal{V})$. We define the position of an alternative j in a preference order \succ in a common way, that is, it is one plus the number of alternatives that are preferred to her:

$$\text{pos}(\succ, j) := |\text{top}(\succ, \{j\})|.$$

2.2 Narcissistic, single-peaked, and single-crossing preferences

Throughout this section, let $\mathcal{P}(\mathcal{V}) = (\succ_1, \succ_2, \dots, \succ_n)$ be a preference profile for the voter set \mathcal{V} .

Narcissistic profiles We call $\mathcal{P}(\mathcal{V})$ a *narcissistic* profile if for each voter $i \in \mathcal{V}$ it holds that she is her most preferred alternative, that is, for each voter $i \in \mathcal{V}$ it holds that $\text{peak}(\succ_i) = i$.

Single-peaked profiles We call $\mathcal{P}(\mathcal{V})$ a *single-peaked* profile if there is a linear order \triangleright over the voter set \mathcal{V} such that each voter is *single-peaked* with respect to \triangleright , that is, for each voter $i \in \mathcal{V}$ and for each two alternatives $a, b \in \mathcal{V}$ it holds that

$$\text{if } a \triangleright b \triangleright \text{peak}(\succ_i) \text{ or } \text{peak}(\succ_i) \triangleright b \triangleright a, \text{ then } b \succ_i a.$$

As already mentioned in the introduction, Example 1 is narcissistic and single-peaked with respect to the order $1 \triangleright 2 \triangleright 3 \triangleright 4$.

There are many equivalent definitions of the single-peaked property. One of them is due to Doignon and Falmagne [12].

Proposition 1 ([12]). *Given a preference profile $\mathcal{P}(\mathcal{V}) = (\succ_1, \succ_2, \dots, \succ_n)$ and a linear order \triangleright on \mathcal{V} , the following statements are equivalent:*

1. $\mathcal{P}(\mathcal{V})$ is single-peaked with respect to \triangleright .
2. For each voter $i \in \mathcal{V}$, and for each alternative $j \in \mathcal{V}$, the top alternatives $\text{top}(\succ_i, \{j\})$ form an interval in \triangleright .

Doignon and Falmagne [12], Escoffier et al. [15] provided polynomial-time algorithms to determine whether a profile is single-peaked. Ballester and Haeringer [1] characterized single-peaked profiles by two forbidden subprofiles:

Proposition 2 ([1]). *A profile is single-peaked if and only if it contains neither a worst-subprofile of three alternatives a, b, c and three voters i, j, k such that*

$$\begin{aligned} \text{voter } i: & \{b, c\} \succ_i a, \\ \text{voter } j: & \{a, c\} \succ_j b, \\ \text{voter } k: & \{a, b\} \succ_k c, \end{aligned}$$

nor an α -subprofile of four alternatives a, b, c, d and two voters i, j such that

$$\begin{aligned} \text{voter } i: & \{a, b\} \succ_i c \succ_i d, \\ \text{voter } j: & \{b, d\} \succ_j c \succ_j a. \end{aligned}$$

Single-crossing profiles We call $\mathcal{P}(\mathcal{V})$ a *single-crossing* profile if there is a linear order \triangleright of the voters \mathcal{V} such that for each pair $\{a, b\} \subseteq \mathcal{V}$ of alternatives and for each three voters $i, j, k \in \mathcal{V}$ with $i \triangleright j \triangleright k$, it holds that

$$\text{if } a \succ_i b \text{ and } a \succ_k b, \text{ then } a \succ_j b.$$

Just as with single-peaked profiles, there are many equivalent definitions of the single-peaked property [12, 6, 7]. To introduce these alternative definitions, let $\text{diff-pairs}(\succ, \succ')$ denote the set of all pairs of alternatives that are ordered differently by \succ and \succ' :

$$\text{diff-pairs}(\succ, \succ') := \{\{i, j\} \subseteq \mathcal{V} \mid i \succ j \text{ and } j \succ' i\}.$$

Proposition 3. [12, 6, 7] Given a preference profile $\mathcal{P}(\mathcal{V}) = (\succ_1, \succ_2, \dots, \succ_n)$ and a linear order \triangleright on \mathcal{V} , the following statements are equivalent:

1. $\mathcal{P}(\mathcal{V})$ is single-crossing with respect to \triangleright .
2. For each pair of alternatives $\{a, b\} \subseteq \mathcal{V}$ and for each two voters $i, j \in \mathcal{V}$ with $i \triangleright j$, it holds that $\text{diff-pairs}(\succ^*, \succ_i) \subseteq \text{diff-pairs}(\succ^*, \succ_j)$, where \succ^* denotes the preference order of the first voter in \triangleright .
3. For each pair $\{a, b\} \subseteq \mathcal{V}$ of alternatives, the voters that prefer a to b and the voters that prefer b to a form an interval in \triangleright , respectively.

Doignon and Falgout [12], Elkind et al. [13], and Brederbeck et al. [6] provided polynomial-time algorithms to determine whether a profile is single-crossing. Brederbeck et al. [6] characterized single-crossing profiles by two forbidden subprofiles:

Proposition 4 ([6]). A profile is single-crossing if and only if it contains neither a γ -subprofile of three pairs of alternatives $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ and three voters i, j, k such that

$$\begin{aligned} \text{voter } i: & \ a \succ_i b \text{ and } c \succ_i d \text{ and } e \succ_i f, \\ \text{voter } j: & \ b \succ_j a \text{ and } d \succ_j c \text{ and } e \succ_j f, \\ \text{voter } k: & \ a \succ_k b \text{ and } d \succ_k c \text{ and } f \succ_k e, \end{aligned}$$

nor a δ -subprofile of two pairs of alternatives $\{a, b\}$ $\{c, d\}$ and four voters i, j, k, ℓ such that

$$\begin{aligned} \text{voter } i: & \ a \succ_i b \text{ and } c \succ_i d, \\ \text{voter } j: & \ b \succ_j a \text{ and } c \succ_j d, \\ \text{voter } k: & \ a \succ_k b \text{ and } d \succ_k c, \\ \text{voter } \ell: & \ b \succ_\ell a \text{ and } d \succ_\ell c, \end{aligned}$$

As mentioned in the introduction, single-crossing narcissistic profiles are a strict subset of single-peaked narcissistic profiles.

Example 2. The profile given in Example 1 is narcissistic, single-peaked, and single-crossing with $1 \triangleright 2 \triangleright 3 \triangleright 4$ being the desired order for single-peakedness and single-crossingness. However, if we change the preference order of voter 3 in Example 1 to obtain the following

$$\begin{aligned} \text{voter 1: } & 1 \succ_1 2 \succ_1 3 \succ_1 4, \\ \text{voter 2: } & 2 \succ_2 3 \succ_2 4 \succ_2 1, \\ \text{voter 3: } & 3 \succ_3 2 \succ_3 1 \succ_3 4, \\ \text{voter 4: } & 4 \succ_4 3 \succ_4 2 \succ_4 1, \end{aligned}$$

then the resulting profile is still narcissistic and single-peaked but not single-crossing anymore. The reason is that the new profile contains subprofiles that are not single-crossing. For instance, it contains a δ -subprofile with respect to the pairs $\{1, 4\}$ and $\{2, 3\}$, and the voters 1, 2, 3, 4.

Proposition 5 ([20]). Every single-crossing narcissistic profile is also single-peaked.

2.3 Semi-Standard Young Tableaux

For a positive integer $n \geq 2$, a *semi-standard Young tableau (SSYT)* of order n [21] consists of n rows of positive integers that satisfy the following:

- i) the i^{th} row has $n - i + 1$ entries with integers between 1 and n , and
- ii) when aligned in the upper-left corner (to obtain an isosceles right triangle), the entries weakly increase along each row and strictly increase down each column.

Example 3 illustrates how an SSYT looks like and shows all eight possible SSYTs of order three.

Remarks The Young tableau [23] was originally defined on a Ferrers diagram which can be of an arbitrary staircase-like shape, that is, it contains n rows of non-increasing length. The numbers in the tableau can be from an arbitrary integer range. For our purpose, it is sufficient to focus on SSYTs for isosceles right triangles and for integer range $[1, n]$. The second condition defined above defines the “semi-standard” property. We refer to the work of Stanley [21], Fulton [16], Yong [22] for further reading.

By Stanley [21]’s hook content lemma, we can derive that the number of SSYTs of order n , denoted as $\#_{\text{SSYT}}(n)$, equals $2^{\binom{n}{2}}$. Before we show this result, we need two more notions.

Definition 1 (Hook lengths and hook contents). Let T be a semi-standard Young tableau of order n . For each i, j with $1 \leq i \leq n$ and $1 \leq j \leq n + 1 - i$, we define the following two notions.

1. The *hook length* of (i, j) , denoted as $h(i, j)$, is one plus the number entries directly below or to the right of $T(i, j)$ in the i^{th} row and j^{th} column: $h_n(i, j) := 2 \cdot (n - i - j) + 3$.
2. The *hook content* of (i, j) is defined as $c_n(i, j) := n - i + j$.

Note that the hook length and the hook content of an SSYT do not depend on the values of the SSYT but on its order. The following example illustrates the concept of SSYTs, together with their hook length and hook content.

Example 3. There are eight different SSYTs of order 3:

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The hook lengths and hook contents of SSYTs of order 3 are:

Hook lengths		
5	3	1
3	1	
1		

Hook contents		
3	4	5
2	3	
1		

Theorem 1. *The number of semi-standard Young tableaux of order n equals $2^{\binom{n}{2}}$.*

Proof. Stanley [21]'s hook content formula states that the number $\#_{\text{SSYT}}(n)$ of SSYTs of order n ($n \geq 2$) equals

$$\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n+1-i}} \frac{c_n(i, j)}{h_n(i, j)}.$$

To show that the above product equals $2^{\binom{n}{2}}$, we show the following:

- i) $\#_{\text{SSYT}}(2) = 2$ and
- ii) for each $n \geq 2$ it holds that $\#_{\text{SSYT}}(n+1) = 2^n \cdot \#_{\text{SSYT}}(n)$.

For the first equation, we can easily check that there are two SSYTs of order two, implying that $\#_{\text{SSYT}}(2) = 2 = 2^{\binom{2}{2}}$.

Now, consider SSYTs of order $n+1$. The definitions of hook lengths and hook contents imply the following equations:

$$\forall i \in \{2, 3, \dots, n+1\}, \forall j \in \{1, 2, \dots, n+2-i\}:$$

$$h_{n+1}(i, j) = 2 \cdot (n+1-i-j) + 3 = h_n(i-1, j), \quad (1)$$

$$c_{n+1}(i, j) = n+1-i+j = c_n(i-1, j). \quad (2)$$

$$\forall j \in \{1, 2, \dots, n-1\}:$$

$$h_{n+1}(1, j) = 2 \cdot (n+1-1-j) + 3 = h_n(1, j-1). \quad (3)$$

$$c_{n+1}(1, j) = n+1-1+j = c_n(1, j+1). \quad (4)$$

By the hook content formula, we can derive the number $\#_{\text{SSYT}}(n+1)$ from the hook lengths and hook contents of SSYTs of order n :

$$\begin{aligned} \#_{\text{SSYT}}(n+1) &= \prod_{\substack{1 \leq i \leq n+1 \\ 1 \leq j \leq n+2-i}} \frac{c_{n+1}(i, j)}{h_{n+1}(i, j)} \\ &\quad \prod_{1 \leq j \leq n+1} \frac{c_{n+1}(1, j)}{h_{n+1}(1, j)} \cdot \prod_{\substack{2 \leq i \leq n+1 \\ 1 \leq j \leq n+2-i}} \frac{c_{n+1}(i, j)}{h_{n+1}(i, j)}. \end{aligned} \quad (5)$$

If we can show that the first factor and the second factor of the product on the right-hand side of equation (5) equals $\#_{\text{SSYT}}(n)$ and 2^n , respectively, then by (5), we can derive that $\#_{\text{SSYT}}(n+1) = 2^n \cdot \#_{\text{SSYT}}(n)$. Thus, it remains to show the following:

$$\prod_{\substack{2 \leq i \leq n+1 \\ 1 \leq j \leq n+2-i}} \frac{c_{n+1}(i, j)}{h_{n+1}(i, j)} = \#_{\text{SSYT}}(n), \text{ and} \quad (6)$$

$$\prod_{1 \leq j \leq n+1} \frac{c_{n+1}(1, j)}{h_{n+1}(1, j)} = 2^n. \quad (7)$$

To show the correctness of equation (6), we use equations (1) and (2):

$$\begin{aligned} \prod_{\substack{2 \leq i \leq n+1 \\ 1 \leq j \leq n+2-i}} \frac{c_{n+1}(i, j)}{h_{n+1}(i, j)} &= \prod_{\substack{2 \leq i \leq n+1 \\ 1 \leq j \leq n+2-i}} \frac{c_n(i-1, j)}{h_n(i-1, j)} \\ &= \prod_{\substack{1 \leq i' \leq n \\ 1 \leq j \leq n+1-i'}} \frac{c_n(i', j)}{h_n(i', j)} = \#_{\text{SSYT}}(n). \end{aligned}$$

The last equality holds by our definition of $\#_{\text{SSYT}}(n)$.

We show equation (7) by induction on n .

First of all, for $n = 2$ we have that $\prod_{1 \leq j \leq n+1} \frac{c_{n+1}(1, j)}{h_{n+1}(1, j)} = \frac{c_3(1,1)}{h_3(1,1)} \cdot \frac{c_3(1,2)}{h_3(1,2)} \cdot \frac{c_3(1,3)}{h_3(1,3)} = \frac{3 \cdot 4 \cdot 5}{5 \cdot 3 \cdot 1} = 4 = 2^2$. Now, suppose that equation (7) holds for $\ell \leq n$, implying that $\prod_{1 \leq j \leq n} \frac{c_n(1, j)}{h_n(1, j)} = 2^{n-1}$. We show that the equality also holds for $\ell := n+1$:

$$\begin{aligned} \prod_{1 \leq j \leq \ell} \frac{c_\ell(1, j)}{h_\ell(1, j)} &= \prod_{1 \leq j \leq n+1} \frac{c_{n+1}(1, j)}{h_{n+1}(1, j)} \\ &= \frac{c_{n+1}(1, n+1) \cdot c_{n+1}(1, n)}{h_{n+1}(1, 1)} \cdot \frac{\prod_{1 \leq j \leq n-1} c_{n+1}(1, j)}{\prod_{2 \leq j \leq n+1} h_{n+1}(1, j)} \\ &\stackrel{\text{def.}}{=} \frac{2 \cdot n + 1}{2 \cdot n + 1} \cdot c_{n+1}(1, n) \cdot \frac{\prod_{1 \leq j \leq n-1} c_{n+1}(1, j)}{\prod_{2 \leq j \leq n+1} h_{n+1}(1, j)} \\ &\stackrel{(3)(4)}{=} c_{n+1}(1, n) \cdot \frac{\prod_{1 \leq j \leq n-1} c_n(1, j+1)}{\prod_{2 \leq j \leq n+1} h_n(1, j-1)} \\ &= \frac{c_{n+1}(1, n)}{c_n(1, 1)} \cdot \frac{\prod_{0 \leq j \leq n-1} c_n(1, j+1)}{\prod_{2 \leq j \leq n+1} h_n(1, j-1)} \\ &\stackrel{\text{def.}}{=} \frac{n+1-1+n}{n-1+1} \cdot \frac{\prod_{1 \leq j' \leq n} c_n(1, j')}{\prod_{1 \leq j' \leq n} h_n(1, j')} \\ &= 2^n \end{aligned}$$

The last equality holds by our induction assumption. \square

3 Counting Single-Peaked Narcissistic Profiles

In this section, we study the number of single-peaked narcissistic (SPN) preference profiles for the voter set $\mathcal{V} = \{1, 2, \dots, n\}$. Recall that a voter is narcissistic if she ranks herself at the first position. Thus,

$$\text{for each } i, \text{ voter } i \text{ has preference order of the form } i \succ_i \dots \quad (8)$$

Since we are interested in the number of different SPN preference profiles (up to renaming), there are always two voters whose preference orders are the reverse of each other.

Lemma 1. *In a single-peaked narcissistic profile $(\succ_1, \succ_2, \dots, \succ_n)$ for voter set \mathcal{V} , there are two voters $i, j \in \mathcal{V}$ such that $|\text{diff-pairs}(\succ_i, \succ_j)| = \binom{n}{2}$.*

Proof. By the characterization of the single-peaked property [1], a single-peaked profile has at most two “worst” alternatives for each triple (see Proposition 2 for more details). If the profile is also narcissistic, then there must be exactly two alternatives a, b that are ranked at the last position by voters b and a , respectively. By (8), the corresponding two voters a and b must satisfy the following:

$$\text{voter } a: a \succ_a \dots \succ_a b \text{ and voter } b: b \succ_b \dots \succ_b a. \quad (9)$$

We claim that the preference orders of voters a and b are the reverse of each other. Suppose for the sake of contradiction that $\text{diff-pairs}(\succ_a, \succ_b)$ does not contain a pair of alternatives $\{c, d\}$, that is, $c \succ_a d$ while $c \succ_b d$. First of all, (9) implies $\{c, d\} \cap \{a, b\} = \emptyset$. Now, the four alternatives a, b, c, d and voters a, b form a forbidden α -subprofile. By Proposition 2, this is a contradiction to the single-peaked property. \square

By Lemma 1, we can rename the voters such that the preference orders of voter 1 and n are the following.

$$\begin{aligned} \text{voter 1: } & 1 \succ_1 2 \succ_1 \dots \succ_1 n, \\ \text{voter } n: & n \succ_n n-1 \succ_n \dots \succ_n 1. \end{aligned} \quad (10)$$

It is easy to show that the only linear orders of alternatives with respect to which voters 1 and n (and thus the whole profile) are single-peaked must be the preference orders of either voter 1 or voter n (also see Lemma 5.1 of the work of Chen et al. [9] for more details). Summarizing, by (10), we can rename the voters such that

$$\text{the single-peaked order of a SPN profile is } 1 \triangleright 2 \triangleright \dots \triangleright n. \quad (11)$$

Example 4. For three voters, we have two different SPN profiles obeying (10):

$$\begin{array}{ll} \text{voter 1: } 1 \succ_1 2 \succ_1 3, & \text{voter 1: } 1 \succ_1 2 \succ_1 3, \\ \text{voter 2: } 2 \succ_2 1 \succ_2 3, & \text{voter 2: } 2 \succ_2 3 \succ_2 1, \\ \text{voter 3: } 3 \succ_3 2 \succ_3 1. & \text{voter 3: } 3 \succ_3 2 \succ_3 1. \end{array}$$

Both profiles are single-peaked with respect to the orders of voters 1 and 3.

Now we are ready to show our main result for the SPN preference profiles.

Theorem 2. *The number of single-peaked narcissistic preference profiles for n voters ($n \geq 2$) is $\prod_{2 \leq i \leq n-1} \binom{n-1}{i-1}$.*

Proof. By (11) and (8), for each voter i , $2 \leq i \leq n-1$, her preference order must satisfy

$$\text{voter } i: i \succ_i i-1 \succ_i \cdots \succ_i 1 \text{ and } i \succ_i i+1 \succ_i \cdots \succ_i n. \quad (12)$$

Thus, if it is clear which positions the alternatives $1, 2, \dots, i-1$ will occupy in the preference order of voter i , then the positions of $i+1, i+2, \dots, n$ are also clear. Then, the preference order of i is also fixed. There are $\binom{n-1}{i-1}$ possible ways to give $i-1$ positions to alternatives $i-1, i-2, \dots, 1$. Altogether, we obtain the desired result for the number of all different SPN profiles (up to renaming). \square

4 Counting Single-Crossing Narcissistic Profiles

In this section, we study the number of single-crossing narcissistic (SCN) preference profiles for the voter set $\mathcal{V} = \{1, 2, \dots, n\}$. Since SCN profiles are also SPN (Proposition 5), we obtain the following result as we did for the SPN profiles.

Proposition 6. *For each single-crossing narcissistic profile for the voter set $\mathcal{V} = \{1, 2, \dots, n\}$, we can rename the voters such that*

- i) *for each $i \in \{1, 2, \dots, n\}$, voter i has preference order $i \succ_i \mathcal{V} \setminus \{i\}$.*
- ii) *voter 1 has preference order $1 \succ_1 2 \succ_1 \cdots \succ_1 n$,*
- iii) *voter n has preference order $n \succ_n n-1 \succ_n \cdots \succ_n 1$,*
- iv) *the profile is single-peaked with respect to $1 \triangleright 2 \triangleright \cdots \triangleright n$, and*
- v) *the profile is single-crossing with respect to $1 \triangleright 2 \triangleright \cdots \triangleright n$ such that the position of each alternative a , $1 \leq a \leq n-1$, along the single-crossing voter order $a+1 \triangleright a+2 \triangleright \cdots \triangleright n$ is non-decreasing.*

Proof. The first four statements follow from Proposition 5 and equations (8), (10), and (11). It remains to show the correctness of the last statement.

Suppose for the sake of contradiction that $1 \triangleright 2 \triangleright \cdots \triangleright n$ is not a single-crossing order. Then, there must be a pair of alternatives $\{a, b\}$ and three voters i, j, k with $i < j < k$ such that $a \succ_i b$ and $a \succ_k b$, but $b \succ_j a$. By (12), it must also hold that $i \succ_i j$, $j \succ_j i$, and $j \succ_k i$, and $j \succ_i k$, $j \succ_j k$, and $k \succ_k j$. Thus, the three pairs $\{a, b\}$, $\{i, j\}$, $\{j, k\}$ and the three voters i, j, k form a forbidden γ -subprofile. By Proposition 4, this implies that the given profile is not single-crossing—a contradiction.

Similarly, if the positions of an alternative a along the voter order $a+1 \triangleright a+2 \triangleright \cdots \triangleright n$ would be decreasing, then there must be an other alternative b and two voters i, j with $a+1 \leq i < j \leq n$ with preferences $b \succ_i a$ and $a \succ_j b$. But since voter a prefers $a \succ_a b$, we obtain that \triangleright is not a single-crossing order—a contradiction. \square

The profile given in Example 2 is evidence that the number of SCN profiles is strictly less than the number of SPN profiles. But how many SCN profiles are there exactly? To answer this question, we first construct a function from SCN profiles with n voters to SSYTs of order $n - 1$. To this end, let \mathbb{P}_n be the set of all SCN profiles with n voters that fulfill the criteria given in Proposition 6 and let \mathbb{S}_{n-1} be the set of all SSYTs of order $n - 1$.

Definition 2 (A binary relation from SCN profiles to SSYTs). Let $f: \mathbb{P}_n \rightarrow \mathbb{S}_{n-1}$ be a function such that for each SCN profile $(\succ_1, \succ_2, \dots, \succ_n) \in \mathbb{P}$ for the voter set $\mathcal{V} = \{1, 2, \dots, n\}$ we obtain an SSYT $f((\succ_1, \succ_2, \dots, \succ_n)) = T$ as follows: For each alternative $i \in \mathcal{V} \setminus \{n\}$, except n , we construct the i^{th} row of T , with $n - i$ entries. Their values depend on the positions of alternative i in the preference orders of voters $n, n-1, \dots, i+1$:

$$\forall j \in \{1, 2, \dots, n - i\}: T(i, j) := n + 1 - \text{pos}(\succ_{n+1-j}, i).$$

Briefly put, the value of T at the i^{th} row and j^{th} column equals the “reverted” position of i in the preference order of voter $n + 1 - j$. The values of each column j are determined by the preference order of voter $n + 1 - j$. The following table gives an illustration of how to build an SSYT T from a given preference profile.

T :

$n + 1 - \text{pos}$	voter n	voter $n - 1$	\dots	voter 2
alter. 1				
alter. 2				
\dots	\dots	\dots		
alter. $n - 1$				

Table 1: An illustration of constructing an SSYT T according to Definition 2.

Note that we do not address the positions of alternative n (in any preference order) since the positions of $1, 2, \dots, n - 1$ determine the position of n . Moreover, the positions of all alternatives in the preference order of voter 1 are also fixed. We use Example 1 to illustrate our function given in Definition 2.

Example 5. Let P denote the profile in Example 1. By Definition 2, the SSYT obtained for P is depicted in the figure below.

T :

1	1	1
2	3	
3		

The positions of alternative 2 in the preference orders of voters 3 and 4 are 2 and 3, respectively. Thus, the second row of $T = f(P)$ has two entries: $T(2, 1) = 4 + 1 - 3 = 2$ and $T(2, 2) = 4 + 1 - 2 = 3$.

Note that the preference order of the last voter is always fixed to $n \succ_n n - 1 \succ_n \dots \succ_n 1$. Indeed, the values of the first column in every SSYT are also fixed, namely $(1, 2, \dots, n - 1)^{n-1}$. Moreover, there are eight such SCN profiles. Our function f will map each of these profiles to a unique SSYT of order 3 given in Example 3.

We show that function f is well-defined and bijective.

Lemma 2. *Function f from Definition 2 is well-defined and bijective.*

Proof. To show that f is well-defined, we need to show that for each given SCN profile $P = (\succ_1, \succ_2, \dots, \succ_n)$ with n voters, $f(P)$ is an SSYT of order $n-1$. That is, we have to show that $T := f(P)$ fulfills the two conditions given in the beginning of Section 2.3.

By Definition 2, $f(P)$ has $n-1$ rows such that the i^{th} row has $(n-1) + 1 - i$ entries. For each $i \in \{1, 2, \dots, n-1\}$ and each $j \in \{1, 2, \dots, n-i\}$, the value $n+1-j$ ranges from n to $i+1$. Thus, $\text{pos}(\succ_{n+1-j}, i)$ is defined and has a value between 2 and n . This means that the value of $T(i, j)$, which is defined as $n+1 - \text{pos}(\succ_{n+1-j}, i)$, is between $n-1$ and 1.

Second, by the last statement in Proposition 6, the positions of the alternatives $i \in \{1, 2, \dots, n-1\}$ in the preference orders of the voters are non-decreasing along the voter order $i+1 \triangleright i+2 \triangleright \dots \triangleright n$. By the double negation in the definition of $T(i, j)$, this implies that the values of along the i^{th} row in T do not decrease.

It remains to show that the values down each column in T strictly increase. This is quite obvious since each column j reflects the positions of the alternatives 1 to $n-j$ in the preference order of voter $n+1-j$. Since these positions strictly decrease and by the negation in the definition of $T(i, j)$, we have that the values down each column in T indeed strictly increase.

Finally, f is a bijection since the definition of $T(i, j)$ is injective and for each SSYT T of order $n-1$, there is an SCN preference profile $(\succ_1, \succ_2, \dots, \succ_n)$:

$$\begin{array}{ccccccc} 1 & : & 1 & \succ_1 & 2 & \succ_1 & \cdots \succ_1 & n \\ 2 & : & 2 & \succ_2 & \cdot & \succ_2 & \cdots \succ_2 & \cdot \\ 3 & : & 3 & \succ_3 & \cdot & \succ_3 & \cdots \succ_3 & \cdot \\ \cdot & : & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ n-1 & : & n-1 & \succ_{n-1} & \cdot & \succ_{n-1} & \cdots \succ_{n-1} & \cdot \\ n & : & n & \succ_n & n-1 & \succ_n & \cdots \succ_n & 1, \end{array}$$

such that for each voter $i \in \{2, \dots, n\}$, her preference order \succ_i is determined by the positions of the alternatives $j \in \{i-1, i-2, \dots, 1\}$, which are $\text{pos}(\succ_i, j) := n+1 - T(j, n+1-i)$. Note that these positions decrease as the variable j increases, which means that we indeed obtain a preference order. Moreover, by the single-peaked property and by the single-peaked order \triangleright , the positions of the alternatives $i+1, i+2, \dots, n$ are fixed once the positions of the alternatives $j \in \{i-1, i-2, \dots, 1\}$ are fixed. \square

Applying the inverse of function f given in Definition 2 on the SSYT produced in Example 5, we will obtain our original profile from Example 1. By Theorem 1 and Lemma 2, we obtain our second main result.

Theorem 3. *The number of single-crossing narcissistic preference profiles for n voters ($n \geq 2$) is $2^{\binom{n-1}{2}}$.*

Proof. Let \mathbb{P}_n be the set of all SCN profiles with n voters that fulfill the criteria given in Proposition 6 and let \mathbb{S}_{n-1} be the set of all SSYTs of order $n-1$. It is clear that both sets are finite. Since the function $f: \mathbb{P}_n \rightarrow \mathbb{S}_{n-1}$ as defined in Definition 2 is a bijection (see Lemma 2), \mathbb{P}_n and \mathbb{S}_{n-1} have the same cardinality. By Theorem 1, we obtain the desired cardinality for \mathbb{P}_n . \square

5 Conclusion

We studied the number of narcissistic profiles that are additionally single-peaked (SPN) or single-crossing (SCN). We established a bijective relation between semi-standard Young tableaux and SCN profiles. By counting the number of semi-standard Young tableaux, we determined the number of SCN profiles. In this paper, we focused on profiles with the same number of voters and alternatives. However, our analysis could be extended to the case where the number of voters is greater than the number of alternatives since the last statement of Proposition 6 still holds in this case and it corresponds to the essential property of an SSYT for an arbitrary Ferrers diagram.

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References

- [1] M. Á. Ballester and G. Haeringer. A characterization of the single-peaked domain. *Social Choice and Welfare*, 36(2):305–322, 2011.
- [2] S. Barberà and B. Moreno. Top monotonicity: A common root for single peakedness, single crossing and the median voter result. *Games and Economic Behavior*, 73(2): 345–359, 2011.
- [3] J. J. Bartholdi III and M. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165–169, 1986.
- [4] D. Black. On the rationale of group decision making. *Journal of Political Economy*, 56(1):23–34, 1948.
- [5] S. J. Brams, M. A. Jones, and D. M. Kilgour. Single-peakedness and disconnected coalitions. *Journal of Theoretical Politics*, 14(3):359–383, 2002.
- [6] R. Bredereck, J. Chen, and G. J. Woeginger. A characterization of the single-crossing domain. *Social Choice and Welfare*, 41(4):989–998, 2013.
- [7] R. Bredereck, J. Chen, and G. J. Woeginger. Are there any nicely structured preference profiles nearby? *Mathematical Social Sciences*, 79:61–73, 2016.
- [8] M.-L. Bruner and M. Lackner. On the likelihood of single-peaked preferences. Technical report, arXiv:1505.05852 [cs.GT], May 2015.
- [9] J. Chen, K. Pruhs, and G. J. Woeginger. The one-dimensional Euclidean domain: Finitely many obstructions are not enough. *Social Choice and Welfare*, pages 1–24, 2016.
- [10] C. H. Coombs. *A Theory of Data*. John Wiley and Sons, 1964.

- [11] G. Demange. Intermediate preferences and stable coalition structures. *Journal of Mathematical Economics*, 23(1):45–58, 1994.
- [12] J. Doignon and J. Falmagne. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms*, 16(2):218–233, 1994.
- [13] E. Elkind, P. Faliszewski, and A. Slinko. Clone structures in voters’ preferences. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC ’12)*, pages 496–513. ACM Press, 2012.
- [14] E. Elkind, M. Lackner, and D. Peters. Preference restrictions in computational social choice: Recent progress. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI ’16)*, pages 4062–4065, 2016.
- [15] B. Escoffier, J. Lang, and M. Öztürk. Single-peaked consistency and its complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence (ECAI ’08)*, pages 366–370. IOS Press, 2008.
- [16] W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.
- [17] S. Karlin. *Total Positivity*. Stanford University Press, 1968.
- [18] J. A. Mirrlees. An exploration in the theory of optimal income taxation. *Review of Economic Studies*, 38:175–208, 1971.
- [19] K. W. Roberts. Voting over income tax schedules. *Journal of Public Economics*, 8(3):329–340, 1977.
- [20] A. Saporiti and F. Tohmé. Single-crossing, strategic voting and the median choice rule. *Social Choice and Welfare*, 26(2):363–383, 2006.
- [21] R. P. Stanley. *Enumerative Combinatorics*, volume 2nd. Cambridge University Press, 1999.
- [22] A. Yong. What is...a Young tableau? *Notices of the American Mathematical Society*, 54(2):240–241, 2007.
- [23] A. Young. On quantitative substitutional analysis. In *Proceedings of the London Mathematical Society*, volume 28, pages 255–292, 1928.